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Effects of jet flow on jet noise via an extension to the Lighthill model

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The Lighthill formalism for jet noise prediction is extended to accommodate wave transport by the mean jet flow. The extended theory combines the simplicity of the Lighthill approach with the generality of the more complex Lilley approach. There is full allowance for 'flow-acoustic' effects: shielding, as well as the refractive 'cone of (relative) silence'. A source term expansion yields a convected wave equation that retains the basic Lighthill source term. This leads to a general formula for power spectral density emitted from unit volume as the Lighthill-based value multiplied by a squared 'normalized' Green's function. The Green's function, referred to a stationary point source, delineates the refraction dominated 'cone of silence'. The convective motion of the sources, with its powerful amplifying effect, also directional, is accounted for in the Lighthill factor. Source convection and wave convection are thereby decoupled, in contrast with the Lilley approach: this makes the physics more transparent. Moreover, the normalized Green's function appears to be near unity outside the 'cone of silence'. This greatly reduces the labour of calculation: the relatively simple Lighthill-based prediction may be used beyond the cone, with extension inside via the Green's function. The function is obtained either experimentally (injected 'point' source) or numerically (computational aeroacoustics). Approximation by unity seems adequate except near the cone and except when there are coaxial or shrouding jets: in that case the difference from unity will quantify the shielding effect. Further extension yields dipole and monopole source terms (cf. Morfey, Mani, and others) when the mean flow possesses density gradients (e.g. hot jets).

1. Introduction

Lighthill, in his seminal papers of 1952 and 1954, posed the problem of flow noise in terms of a wave equation for a 'uniform medium at rest, which coincides with the real fluid outside the region of flow'. The actual flow was effectively incorporated into right-hand-side terms, which were interpreted as sources of sound. Although the equation is exact, approximations to these source terms have the effect of suppressing sound convection (hence refraction and shielding) by the mean jet flow (Ribner 1962, 1964). (Some effects of refraction were pointed out by Powell (1954) even before this connection was made.) Equivalent equations for a moving medium have been put forward (e.g. Phillips 1960; Csanady 1966; Schubert 1969, 1972*a*, *b*; Lilley 1972): they allow for the sound convection. Of these, Lilley's equation has received much attention: it has been developed by Mani (1972, 1976*a*, *b*), Balsa (1976*a*, *b*, 1977), and others (Tester & Morfey 1976; Morfey, Szewczyk & Tester 1978; Balsa & Gliebe 1977; Balsa *et al.* 1978, Gliebe & Balsa 1978; Gliebe *et al.* 1991) into a quantitative predictive formalism for properties of jet noise. It entails a formidable derivation or calculation of a Green's function for a moving source in highly idealized models of a jet flow. By contrast, the Lighthill procedure is relatively simple, as developed by Ribner (1969), Pao & Lowson (1970), and others based on Csanady (1966) (Krishnappa 1968; Krishnappa & Csanady 1969; Moon & Zelazny 1975). These various formalisms appear to yield comparable predictive accuracy outside the 'cone of silence' (figure 1) opening downstream of the jet (Ribner 1977, 1981). Within this cone the Lighthill-based theory fails completely – it predicts no attenuation (figure 1, pattern $B+C$) – whereas the Lilley-based theory exhibits good to poor accuracy, depending on frequency (see Mani 1976*a*).

Certain early analytical findings (see below) led to the notion that the Lighthill formalism could be extended so as to be valid both inside and outside the 'cone of silence' like Lilley's. The end result would be attractive: the extended Lighthill theory would have the simplicity of the Lighthill approach and the generality of the Lilley approach. There would be full allowance for 'flow-acoustic' effects, shielding as well as the refractive 'cone'. That development has been the object of the present paper. We turn now to the underlying concepts.

It is now well known (Ribner 1962, 1964; Csanady 1966) that expansion of the basic Lighthill source term leads to extra terms that may be shifted to the left-hand side to yield a convected wave equation: it was the implicit discarding of these extra terms that was cited further above. The expansion was exploited in Ribner (1977, 1981) to demonstrate a considerable equivalence between the Lighthill- and Lilley-based approaches outside the 'cone of silence'. In the course of the present study it was realized that the dominant residual source terms, in the 'parallel flow' approximation, were fully equivalent to the Lighthill term.

The basis of the extension lies in replacing the ordinary oscillatory Green's function, $e^{ikr}/4\pi r$, by that for the convected wave equation. These are both for a stationary point source, in contradistinction to the moving source of the Lilley procedures; this decouples convection of the sources and the sound waves, permitting the cited simplifications in the theory. In consequence, the physical interpretation is also more transparent. The following analysis develops the mathematical implementation. Since the Green's function is frequency-dependent, the formalism is directed toward the power spectral density, in particular, from unit volume. Then integrations can provide the full jet spectrum and the broadband noise intensity. All of these are direction-dependent.

The present study is based on Ribner (1995) with both condensation and revision: it evolved from the survey paper, Ribner (1981). The earlier paper illuminates certain facets only briefly touched on herein; moreover, it displays graphically several comparisons of theory and experiment merely cited here. For a fuller understanding and perspective as to how the present notions relate to other theories of jet noise, that paper should be consulted as well.

2. Modified Lighthill equation

2.1. *Unconvected wave equation: virtual medium at rest*

Lighthill manipulated the conservation equations of fluid dynamics into the form of a wave equation forced primarily by nonlinear terms in the unsteady velocity v_i on the right-hand side. When pressure replaces the density that he used as independent variable his equation takes the form

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p=\frac{\partial^2 \rho v_i v_j}{\partial x_i \partial x_j}+\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 \rho}{\partial t^2}. \tag{1}$$

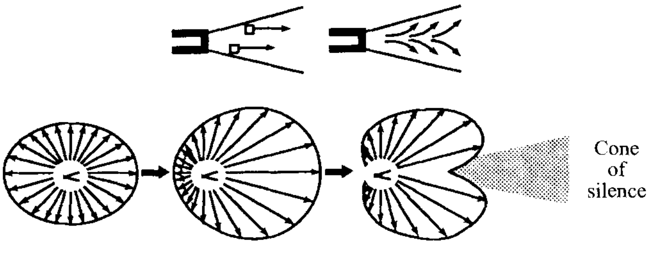

FIGURE 1. Synthesis of the directional pattern of jet noise, displaying the refractive zone: the 'cone of silence'. Logarithmic (decibel) scale. B, basic pattern; C, convective amplification; R, refraction.

Additionally, the fluid has been approximated as inviscid (justified in Lighthill 1952, Goldstein 1976). The right-hand side is interpreted as a spatial distribution of sources of sound. All of the effects of the flow – the turbulence and the mean flow – are incorporated in the source terms. They are treated as if imbedded in a 'virtual medium at rest'. For the fluid as here idealized, (1) is exact.

In the usual approximation, the last two terms are taken to cancel, and the fluid density is taken to be a constant ($\rho = \rho_0$) in the first. This leads to

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p = \rho_0 \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j}. \tag{2}$$

A rich literature (much of it cited in Ribner 1964, 1981) has dealt with applications of this equation for the prediction of properties of jet noise.

It is implicit in (2) that the time-average density is spatially uniform ($\bar{\rho} = \rho_0$). Important additional source terms arise when $\bar{\rho}$ is non-uniform, as in a heated jet or jet of foreign gas (Morfey 1973; Mani 1976*b*; Michalke & Michel 1979, 1980). Their derivation is included in the Appendix.

The early replacement of ρ by ρ_0 to yield (2) is premature, however; it has the effect of suppressing wave convection (and refraction) by the flow (Ribner 1962, 1964). The first step in the demonstration is an expansion of the original first source term. It takes the simplest form under the specification of a unidirectional, transversely sheared, mean flow $U(x_2)$. The instantaneous local velocity is written as the mean plus a perturbation u_i,

$$v_i = U_i + u_i, \quad U_i = (U(x_2), 0, 0) \tag{3}$$

and the expansion changes (1) to

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U}{\partial x_2}\frac{\partial \rho u_2}{\partial x_1} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\bar{\mathrm{D}}^2 \rho}{\mathrm{D}t^2} \tag{4}$$

using the definition

$$\bar{\mathrm{D}}/\mathrm{D}t = \partial/\partial t + U\partial/\partial x_1 \tag{5}$$

as a convective derivative following the mean flow. Equation (4) is restricted compared with (1) by the 'parallel flow' specification – a local approximation to a jet flow; for this scenario it is exact.

2.2 *Convected wave equation: actual medium with flow*

At this point we approximate $\bar{D}^2\rho/Dt^2$ as $\bar{c}^{-2}\bar{D}^2p/Dt^2$. Where $\bar{c} = \bar{c}(\boldsymbol{x})$ is a local time-average sound speed (the order of the error is examined in Ribner 1995, Appendix B). On shifting the term to the left-hand side, (4) becomes

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2p}{Dt^2} - \nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U}{\partial x_2}\frac{\partial \rho u_2}{\partial x_1}. \tag{6}$$

This is not yet in final form. The second source term, involving mean flow shear $\partial U/\partial x_2$, is linear in u_2; this implies a linear connection with p via the momentum equation (see below). For this reason, it has been argued (Schubert 1969, 1972; Lilley 1972; Doak 1972) that the term participates in wave propagation and so should be on the left-hand side. (See Goldstein 1976 for a further discussion.) This applies, however, only to a small acoustic (or compressible) component associated with wave propagation. Within a subsonic flow the overwhelming part of u_i is induced by the turbulence vorticity; being small compared with the soundspeed, it may be approximated as incompressible. Thus we split off the small acoustic component of the shear term (containing u_{2ac}) and place it on the left-hand side. But we neglect its loss from the right-hand side as being negligible. (The acoustic component of the first source term, on the other hand, is of higher order and may be neglected on both sides.)

It is at this point that we may justifiably apply Lighthill's approximation $\rho = \rho_0$ to the remaining right-hand-side source terms. Consistently, the u_i are taken to have zero divergence. Equation (6) then becomes

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2p}{Dt^2} - 2\rho_0\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \rho_0\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\rho_0\frac{\partial U}{\partial x_2}\frac{\partial u_2}{\partial x_1}, \tag{7}$$

where u_{2ac} is defined via the momentum equation

$$\rho_0\frac{\bar{D}u_{2ac}}{Dt} = -\frac{\partial p}{\partial x_2}. \tag{8}$$

The corresponding equation for an oscillatory point source is

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2p}{Dt^2} - 2\rho_0\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \delta(\boldsymbol{x}-\boldsymbol{y})\,e^{-i\omega t}. \tag{9}$$

The solution of the pair of equations (8) and (9) for p may be written $G\,e^{-i\omega t}$, defining the Green's function $G(\boldsymbol{x},\boldsymbol{y}\,|\,\omega)$ in the frequency domain. The Green's function will be central to the further developments herein.

In applications, we specialize to the far-field asymptotic form of the Green function, G, decaying like $1/r$, $r \equiv |\boldsymbol{x}-\boldsymbol{y}|$. Then, compatibly, the pressure p of (8) refers to an acoustic field; within the flow it is weak compared with the 'pseudosound' pressure field (Blokhintsev 1956) of the turbulence. The pseudosound pressure depends on the excluded near-field terms of G that decay like $1/r^n$.

Recall now that (4)–(9) relate to turbulence u_i superposed on a transversely sheared mean flow $u_i = (U(x_2), 0, 0)$. For this scenario we examine the Lighthill source term $\rho_0\,\partial v_i v_j/\partial x_i x_j$ of (2), where $v_i = U_i + u_i$. On carrying out the differentiation, the term expands exactly into the two terms on the right-hand side of (7). Thus that equation is exactly equivalent to

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2p}{Dt^2} - 2\rho_0\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \rho_0\frac{\partial^2 v_i v_j}{\partial x_i \partial x_j}. \tag{10}$$

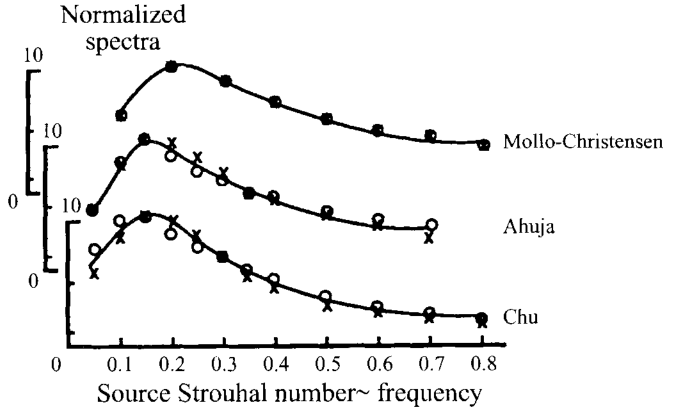

FIGURE 2. Shear term in postulated source role. Comparison of experimentally derived ×, shear noise and ○, self-noise spectra, normalized to the same amplitude. The self-noise has been downshifted one octave, in accordance with the theory. The close match supports the rationale. (After Ribner 1981, based on Nossier & Ribner 1975.)

In the expansion we have used the incompressibility relation $\partial v_i/\partial x_i = 0$ implied by taking $\rho = \rho_0$; as noted, the approximation of incompressibility is made only in the source terms.

Equation (10) is a major new result: it generalizes the unconvected wave equation (2) of Lighthill to allow explicitly for the mean jet flow. By accounting for wave convection, its solutions will display sound refraction by velocity gradients. An appealing feature is the retention of the same right-hand-side source term as in (2). This is traceable to the deferral of the step $\rho = \rho_0$ in the term: the degree of approximation in making that step between (6) and (7) is less than in making it between (1) and (2).

According to Lighthill's physical arguments, $\rho_0 \partial v_i v_j/\partial x_i x_j$ of (10) is a valid source term for flow noise. From the above it is clear that the two right-hand source terms of (7), being an equivalent expansion for the sheared mean flow, are likewise valid source terms for that scenario: the 'shear noise' term $2\rho_0(\partial U/\partial x_2)(\partial u_2/\partial x_1)$ as well as the 'self-noise' term, $\rho_0 \partial u_i u_j/\partial x_i x_j$. In an alternative development, Lilley, in effect, moved virtually the entire shear term to the left-hand-side wave operator (Lilley 1972; Mani 1976*a*). The 'shear noise' source term so lost from the right-hand side is of major importance.

This was shown indirectly by Ribner (1964, 1969), and later more directly by Pao & Lowson (1970) in terms of Lighthill's equation. In their analyses, they postulated isotropic turbulence superposed on a sheared mean flow. For this scenario the radiation from the first source term of (7) ('self-noise' from turbulence alone) is omnidirectional, whereas that from the second source term ('shear noise') has a dipole-like directivity. Thus the combined directivity (the 'basic' directivity of the source pattern) is somewhat ellipsoidal (figure 1). Moreover, if u_i is proportional to $e^{-i\omega t}$, then $u_i u_j$ is proportional to $e^{-i2\omega t}$, a frequency doubling. This suggests that the self-noise spectrum lies roughly an octave above the shear-noise spectrum. Because of the differences in directivity, the two spectra (in terms of source frequency) can be separated out from experimental measurements: they can be extracted from a pair of readings at $\vartheta = 45°$ and $\vartheta = 90°$ at each frequency (Nossier & Ribner 1975). Some results from data of several experimenters are plotted in figure 2: self-noise spectra, downshifted one octave, are superposed on shear-noise spectra. These show a surprisingly good match. Thus, not only does the shear noise exist, but its spectrum is related to the self-noise spectrum in a fashion implied by the theory. (It is noted that Lilley's formalism contains a residual 'shear-noise' source term, defined differently. It remains after the shift of the major part to the left hand side. Being of order turbulent velocity squared, its noise spectrum will not be downshifted an octave from the self-noise spectrum.)

We reiterate the restrictive assumption underlying the above results: they are based on the specification of a unidirectional, transversely sheared, mean flow $U(x_2)$, of uniform density. This limitation is partly relaxed in the Appendix in the context of an exact expansion of the Lighthill source term. This employs a general flow $U(\boldsymbol{x})$ and then introduces approximations appropriate to a jet flow. It supports the applicability of (7) and (10) to realistic spreading jets (with the subscript 2 in the shear term referring to the radial direction in a round jet). The expansion, moreover, allows for mean flow density gradients; these give rise to important additional source terms in the cases of hot jets and jets of foreign gases (Morfey 1973; Mani 1976*b*; Michalke & Michel 1979, 1980).

The following sections develop a formalism for the power spectral density of the sound radiated by the source terms on the right hand side of the convected wave equation (10).

3. Formulae for power spectral density

3.1. *Arbitrary source term, $Q(\boldsymbol{x}, t)$*

The formulae that follow are based on the Green function $G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)$ for a stationary oscillatory point source at point $\boldsymbol{y}$. The approach parallels that of Balsa (1977: Appendix) based on a moving oscillatory point source, with missing steps being inferred. We seek the power spectral density $\Phi(\boldsymbol{x} \mid \omega)$ of the radiated sound pressure dictated by equations (2) or (10); they may be written symbolically as

$$\mathscr{L}[\partial/\partial t, \partial/\partial \boldsymbol{x}; \boldsymbol{a}(\boldsymbol{x})]\, p(\boldsymbol{x}, t) = Q(\boldsymbol{x}, t), \tag{11}$$

where $\mathscr{L}$ may be either the unconvected wave operator of (2) or the convected wave operator of (10); $Q(\boldsymbol{x}, t)$ is an arbitrary source term. The $\boldsymbol{a}(\boldsymbol{x})$ are the coefficients; for the convected wave operator they allow for the local mean flow, taken as $U(x_2)$, and a space-variable sound speed. (In a generalized version of (10) in Appendix A, $U(x_2)$ becomes $U(\boldsymbol{x})$, and ρ_0 becomes $\bar{\rho}(\boldsymbol{x})$, where $\bar{\rho}$ is a local time average. Equation (11) applies to this version as well.) Correspondingly,

$$\mathscr{L}[-\mathrm{i}\omega, \partial/\partial \boldsymbol{x}; \boldsymbol{a}]\, \hat{p}(\boldsymbol{x} \mid \omega) = \hat{Q}(\boldsymbol{x} \mid \omega) \tag{12}$$

where p, $\hat{p}$ and Q, $\hat{Q}$ are defined in Fourier transform pairs:

$$\hat{p}(\boldsymbol{x} \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\boldsymbol{x}, t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t, \quad p(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \hat{p}(\boldsymbol{x} \mid \omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega, \tag{13}$$

$$\hat{Q}(\boldsymbol{x} \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\boldsymbol{x}, t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t, \quad Q(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \hat{Q}(\boldsymbol{x} \mid \omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{14}$$

Equation (12) has a solution

$$\hat{p}(\boldsymbol{x} \mid \omega) = \int_{-\infty}^{\infty} G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)\, \hat{Q}(\boldsymbol{y} \mid \omega)\, \mathrm{d}^3 \boldsymbol{y} \tag{15}$$

in terms of a Green's function $G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)$ in the frequency domain that is the solution of

$$\mathscr{L}[-\mathrm{i}\omega, \partial/\partial \boldsymbol{x}; \boldsymbol{a}]\, G(\boldsymbol{x}, \boldsymbol{y} \mid \omega) = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{16}$$

The (two-sided) power spectral density of the sound pressure is evaluated as

$$\Phi(\boldsymbol{x} \mid \omega) = \langle \hat{p}(\boldsymbol{x} \mid \omega)\, \hat{p}^*(\boldsymbol{x} \mid \omega) \rangle \tag{17}$$

where $\langle\ \rangle$ signifies an ensemble average. Inserting (15), with $\boldsymbol{y}$ replaced by $\boldsymbol{y}'$ and $\boldsymbol{y}''$,

$$\Phi(\boldsymbol{x} \mid \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\boldsymbol{x}, \boldsymbol{y}' \mid \omega)\, G^*(\boldsymbol{x}, \boldsymbol{y}'' \mid \omega) \langle \hat{Q}(\boldsymbol{y}' \mid \omega)\, \hat{Q}^*(\boldsymbol{y}'' \mid \omega) \rangle\, \mathrm{d}^3 \boldsymbol{y}'\, \mathrm{d}^3 \boldsymbol{y}'' \tag{18}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{\xi}/2 \mid \omega)\, G^*(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}/2 \mid \omega) \langle \hat{Q}(\boldsymbol{y} + \boldsymbol{\xi}/2 \mid \omega)\, \hat{Q}^*(\boldsymbol{y} - \boldsymbol{\xi}/2 \mid \omega) \rangle\, \mathrm{d}^3 \boldsymbol{\xi}\, \mathrm{d}^3 \boldsymbol{y}, \tag{19}$$

where $\boldsymbol{\xi} = \boldsymbol{y}' - \boldsymbol{y}''$ and $\boldsymbol{y} = (\boldsymbol{y}' + \boldsymbol{y}'')/2$.

3.2. *Reduction for far field*

The Green's function of (19) may be written in the form

$$G(\boldsymbol{x}, \boldsymbol{y} \mid \omega) = |G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)|\, \mathrm{e}^{\mathrm{i}\psi(\boldsymbol{x}, \boldsymbol{y} \mid \omega)}. \tag{20}$$

We now restrict $\boldsymbol{x}$ to the far field defined by $|\boldsymbol{x}|$ being very much greater than both $|\boldsymbol{y}|$ and the largest wavelengths of concern; $\boldsymbol{y}$ is limited to the region of non-zero source strength Q. A sufficient approximation for the phase, which seems to be implied by Balsa (1977), is then

$$\psi(\boldsymbol{x}, \boldsymbol{y}' \mid \omega) = \psi(\boldsymbol{x}, \boldsymbol{y}'' \mid \omega) - \boldsymbol{\kappa} \cdot \boldsymbol{\xi} \tag{21}$$

(see after (19)) where the wave vector

$$\boldsymbol{\kappa} = -(\boldsymbol{\nabla}_y \psi)_{far\ field} \tag{22}$$

is proportional to ω. There seems to be the further reasonable assumption, which we make also, that the variation in amplitude of G is negligible compared with that of the phase as $\boldsymbol{\xi}$ of (19) ranges within the source region Q. Then $\boldsymbol{\xi}$ may be dropped in comparison with $\boldsymbol{y}$ in the amplitude so that

$$G(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{\xi}/2 \mid \omega)\, G^*(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{\xi}/2 \mid \omega) \approx |G(\boldsymbol{x}, \boldsymbol{y} \mid w)|^2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\xi}}. \tag{23}$$

The other factor in (19) is the frequency-domain correlation

$$\hat{R}(\boldsymbol{y}, \boldsymbol{\xi} \mid \omega) \equiv \langle \hat{Q}(\boldsymbol{y} + \boldsymbol{\xi}/2 \mid \omega)\, \hat{Q}^*(\boldsymbol{y} - \boldsymbol{\xi}/2 \mid \omega) \rangle. \tag{24}$$

This is the Fourier transform of the time-domain correlation

$$R(\boldsymbol{y}, \boldsymbol{\xi}, \tau) = \langle Q(\boldsymbol{y} + \boldsymbol{\xi}/2, t + \tau)\, Q(\boldsymbol{y} - \boldsymbol{\xi}/2, t) \rangle \tag{25}$$

(which is independent of t); specifically,

$$\hat{R}(\boldsymbol{y}, \boldsymbol{\xi} \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\tau} R(\boldsymbol{y}, \boldsymbol{\xi}, \tau)\, \mathrm{d}\tau. \tag{26}$$

Insertion of (23) and (26) into (19) yields

$$\Phi(\boldsymbol{x} \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)|^2\, \mathrm{d}^3 \boldsymbol{y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\boldsymbol{y}, \boldsymbol{\xi}, \tau)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\xi} + \mathrm{i}\omega\tau}\, \mathrm{d}^3 \boldsymbol{\xi}\, \mathrm{d}\tau. \tag{27}$$

Equation (27) is the desired general result for the power spectral density $\Phi(\boldsymbol{x} \mid \omega)$. The inner integral can be recognized as a four-dimensional Fourier transform of the two-point space–time correlation $R(\boldsymbol{y}, \boldsymbol{\xi}, \tau)$ of the source pattern $Q(\boldsymbol{y}, t)$. Alternatively, it is a three-dimensional transform of the cross-spectral density $R(\boldsymbol{y}, \boldsymbol{\xi} \mid \omega)$. This transform multiplied by the square of the amplitude of the Green's function (frequency domain) has a simple interpretation: it is the contribution of unit volume of the sources Q to the power spectral density of the sound pressure radiated to the field point $\boldsymbol{x}$.

3.3. *Moving reference frame*

Experimentally the correlation $R(\boldsymbol{y}, \boldsymbol{\xi}, \tau)$ in a jet flow has a form describing a moving, fluctuating pattern. This is dealt with most neatly by transforming to a reference frame moving with the pattern convection velocity, taken as U_c (but the Green's function, unlike that in the formalism of Balsa 1977, still refers to a point source at rest). Following Chu (1966, 1973), we take

$$\boldsymbol{\xi}_m = \boldsymbol{\xi} - \boldsymbol{U}_c \tau, \quad \boldsymbol{U}_c = (U_c, 0, 0), \tag{28}$$

so that

$$\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_m = \boldsymbol{\kappa} \cdot \boldsymbol{\xi} - \boldsymbol{\kappa} \cdot \boldsymbol{U}_c \tau, \tag{29}$$

and re-express R in terms of $\boldsymbol{\xi}_m$, using (28), as

$$R_m(\boldsymbol{y}, \boldsymbol{\xi}_m, \tau) = R(\boldsymbol{y}, \boldsymbol{\xi}, \tau). \tag{30}$$

Then in (27)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\boldsymbol{y}, \boldsymbol{\xi}, \tau) (\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\xi} + \mathrm{i}\omega\tau} \, \mathrm{d}^3 \boldsymbol{\xi} \, \mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_m(\boldsymbol{y}, \boldsymbol{\xi}_m, \tau) (\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_m + \mathrm{i}(\omega - \boldsymbol{\kappa} \cdot \boldsymbol{U}_c)\tau} \, \mathrm{d}^3 \boldsymbol{\xi}_m \, \mathrm{d}\tau \tag{31}$$

since the Jacobian of the transformation $\boldsymbol{\xi} \rightarrow \boldsymbol{\xi}_m$ is unity. We may further define

$$\bar{\omega} = \omega - \boldsymbol{\kappa} \cdot \boldsymbol{U}_c \tag{32}$$

as the effective Doppler shifted source frequency in the moving frame to yield an observer frequency ω at $\boldsymbol{x}$ in the stationary frame (far field). Inserting these last two equations converts (27) into

$$\Phi(\boldsymbol{x} \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)|^2 \, \mathrm{d}^3 \boldsymbol{y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_m(\boldsymbol{y}, \boldsymbol{\xi}_m, \tau) \, \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_m + \mathrm{i}\bar{\omega}\tau} \, \mathrm{d}^3 \boldsymbol{\xi}_m \, \mathrm{d}\tau. \tag{33}$$

This is an alternative form for the power spectral density in which R is re-expressed (as R_m) in a moving reference frame: it is more useful in that the effects of source convection are more easily brought out. In (27) the space–time source field correlation is referred to a stationary coordinate frame and is designated R. In (33) this same correlation is referred by transformation to a coordinate frame moving with velocity U_c, and is designated R_m. In both cases the two points being correlated are stationary.

3.4. *Lighthill source term,* $Q(\boldsymbol{x}, t) = \bar{\rho}(\boldsymbol{x}) \, \partial^2 v_i, v_j / \partial x_i \, \partial x_j$

Consider a source strength distribution of the form of the right-hand side of (10). When generalized (Appendix) to allow for space-variable time-average density (ρ_0 replaced by $\bar{\rho}(\boldsymbol{x})$) it takes the form

$$Q(\boldsymbol{x}, t) = \bar{\rho}(\boldsymbol{x}) \, \partial^2 v_i v_j / \partial x_i \, \partial x_j. \tag{34}$$

An extension of the derivation leading to (27) can be carried out (Ribner 1995) wherein the differentiation leads to factors k_i. The power spectral density of the sound radiated by $Q(y,t)$ of this functional form comes out to be

$$\Phi(x \mid \omega) = \int_{\infty}^{\infty} |G(x,y|\omega)|^2 \bar{\rho}^2(y) \times \kappa_i \kappa_j \kappa_k \kappa_l \langle v_i v_j(y', t+\tau) v_k v_l(y'', t)\rangle \, \mathrm{e}^{-\mathrm{i}\kappa\cdot\xi+\mathrm{i}\omega\tau} \, \mathrm{d}\tau \, \mathrm{d}^3\xi \, \mathrm{d}^3 y. \qquad (35)$$

On taking $\kappa_i \kappa_j \kappa_k \kappa_l$ inside the $\langle\ \rangle$ of (35), we have terms like $\kappa_i \kappa_j v_i v_j$. But the summation $\kappa_i v_j$ is κ times the component of $\boldsymbol{v}$ along κ, which we designate v_κ. Writing κ as the magnitude of $\boldsymbol{\kappa}$, the summation represented by this term as i, j, k, l range from 1 to 3 reduces to

$$\kappa_i \kappa_j v_i v_j = \kappa^2 v_\kappa^2. \qquad (36)$$

(Notice the parallel with the result obtained by Proudman 1952 for a virtual medium at rest,

$$x_i x_j v_i v_j = x^2 v_x^2, \qquad (37)$$

which he used to simplify the Lighthill far field solution.) It follows that, in (35), the summation

$$\kappa_i \kappa_j \kappa_k \kappa_l \langle v_i v_j(y', t+\tau) v_k v_l(y'', t)\rangle = \kappa^4 \langle v_\kappa^2(y', t+\tau) v_\kappa^2(y'', t)\rangle \equiv \boldsymbol{\kappa}^4 R_\kappa(y, \xi, \tau), \qquad (38)$$

where R_κ is a two-point correlation of v_κ^2. With this replacement (35) simplifies to

$$\Phi(x \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \kappa^4 \bar{\rho}^2(y) \, | G(x, y \mid \omega)|^2 \, \mathrm{d}^3 y \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_\kappa(y, x, \tau) \, \mathrm{e}^{-\mathrm{i}\kappa\cdot\xi+\mathrm{i}\omega\tau} \, \mathrm{d}^3\xi \, \mathrm{d}\tau, \qquad (39)$$

where it is noted that the two points being correlated in R_κ are referred to a stationary reference frame. Correspondingly, (33) becomes

$$\Phi(x \mid \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \kappa^4 \bar{\rho}^2(y) \, | G(x, y \mid \omega)|^2 \, \mathrm{d}^3 \boldsymbol{v} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\kappa m}(y, \xi_m, \tau) \, \mathrm{e}^{-\mathrm{i}\kappa\cdot\xi_m+\mathrm{i}\bar{\omega}\tau} \, \mathrm{d}^3\xi_m \, \mathrm{d}\tau, \qquad (40)$$

where the correlation $R_{\kappa m}$ designates the same correlation with respect to the moving frame (velocity U_c); it is obtained from R_κ via the transformation (31). Equations (39) and (40) are key results.

These latest formulations for the power spectral density $\Phi(x \mid \omega)$ may be put in perspective: they are all expressed in terms of the Green's function $G(x, y \mid \omega)$ for a stationary, oscillatory point source in an arbitrary flow $U(x)$. Equations (27) and (33) refer to a general source strength function Q; (39) and (40), on the other hand, refer to a source strength of the generalized Lighthill form $Q = \bar{\rho}(y) \, \partial^2 v_i v_j / \partial y_i \, \partial y_j$ (summed), to which $\bar{\rho}(y) \, \kappa^2 v_\kappa^2$ is equivalent.

For exact application to (7), (10), (A 12), and (A 14), $U(x)$ takes the restricted form of a transversely sheared flow $U(x_2)$. The Appendix (see (A 15) and (A 16)) justifies a relatively weak additional variation of U with axial distance x_1, and it replaces x_2 with radial distance r for a round jet. Using this approximation, the Green's function $G(\boldsymbol{x},\boldsymbol{y}\,|\,\omega)$ herein refers to a realistic spreading round jet. (On the other hand, the Green's function of the Lilley-based procedures has always been evaluated for a non-spreading cylindrical jet.)

It is noteworthy (Lighthill 1952) that this double divergence form of the term Q implies that the sound sources are of quadrupole nature. In the time domain this was associated with an operator $\partial^2/\partial t^2$ in the far-field format; in the present wavenumber domain the factor κ^2 plays an equivalent role in the final equations.

3.5. *Virtual medium at rest*

This is the scenario of Lighthill (1952, 1954): the mean fluid flow U_i is incorporated into the source term via $v_i = U_i + u_i$. The original region of flow is now treated as a 'virtual medium at rest': U_i is effectively zero in the left-hand-side wave operator. For this case the oscillatory Green's function is simply

$$G(\boldsymbol{x},\boldsymbol{y}\,|\,\omega) = \frac{1}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{e}^{\mathrm{i}k\,|\boldsymbol{x}-\boldsymbol{y}|}, \quad k = w/c_0. \tag{41}$$

When the observer point $\boldsymbol{x}$ is in the far field (cf. after (20)), a sufficient approximation is, with $x = |\boldsymbol{x}|$,

$$G(\boldsymbol{x},\boldsymbol{y}\,|\,\omega) = \frac{1}{4\pi x}\,\mathrm{e}^{\mathrm{i}kx - \mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{y}/x}, \quad G(\boldsymbol{x},\boldsymbol{y}\,|\,\omega) = \frac{1}{4\pi x}. \tag{42}$$

Thus, in this case, the vector $\boldsymbol{\kappa} = -(\boldsymbol{\nabla}_y\,\psi)_{far\,field}$ of (22) may be identified with the wave vector $\boldsymbol{k}$ given by

$$\boldsymbol{\kappa} = \boldsymbol{k} = k\boldsymbol{x}/x = \omega\boldsymbol{x}/c_0 x, \quad k = |\boldsymbol{k}|. \tag{43}$$

Also, with the sources convected parallel to the x_1-axis, $\boldsymbol{U}_c = (U_c, 0, 0)$, (32) yields

$$\bar{\omega} = \omega[1-(\boldsymbol{U}_c/c_0)\cdot(\boldsymbol{x}/x)] = \omega(1-M_c\cos\theta) \equiv \omega\Theta \tag{44}$$

This is just the Doppler-shifted source frequency that will yield an observed frequency ω at $\boldsymbol{x}$.

3.6. *Compact eddy approximation*

The time delay τ_m^* between correlated source points is related to $\boldsymbol{k}\cdot\xi_m$ by the equation

$$\omega\Theta\tau_m^* = \boldsymbol{k}\cdot\xi_m. \tag{45}$$

This is inferred from Chu (1973, above his equation (8), employing our (43) and (44)). Quoting from him, '...According to Lighthill, retarded time can be neglected if $\omega L/c_0$ is small so that the eddy size L is small compared with the wave-length of the sound that it generates. If this condition is met... then for $\xi_m \leqslant L$ the term $\cos\omega\Theta\tau_m^*$ [and similarly, $\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\xi_m}$] can be approximated as unity'. When the turbulent 'eddies' meet this condition, the space–wavenumber transform of R_{km} of (40) (using $\boldsymbol{\kappa} = \boldsymbol{k}$) becomes merely a volume integral. Thus the simplifying assumption that the 'eddies' are 'acoustically compact' in this sense permits a marked simplification. Most formulations for quantitative prediction of jet noise implicitly exploit this approximation. Unfortunately, it leads to overprediction of the amplifying effect of source convection at high subsonic speeds. We will return to this point later.

3.7. *Actual medium vs. virtual medium*

Here we compare the noise emission from the actual medium, allowing for the effect of the fluid flow on propagation, to that predicted for the virtual medium at rest. We will show how the former differs from the latter (the Lighthill scenario) in being an extension to allow for flow–acoustic interaction effects: e.g. refraction that bends sound rays away from the jet axis to create a 'cone of silence' opening downstream, and we will show a close approach to the Lighthill case outside the cone of silence.

It will be convenient to restrict attention to the power spectral density emitted from unit volume at y. We designate this $\Phi(x, y \mid \omega)$ and define it by the equation

$$\Phi(x \mid \omega) = \int_{-\infty}^{\infty} \Phi(x, y \mid \omega) \, \mathrm{d}^3 y. \tag{46a}$$

We apply this to reinterpret equation (40) ('actual medium'), with the appropriate Green's function and with restriction to uniform mean density ($\bar{\rho}(y) = \rho_0$). For that scenario

$$\Phi(x, y \mid \omega) = \rho_0^2 k^4 |G(x, y \mid \omega)|]^2 \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{km}(y, \xi_m, \tau) \, \mathrm{e}^{-\mathrm{i}\kappa \cdot \xi_m + \mathrm{i}\omega\tau} \, \mathrm{d}^3 \xi_m \, \mathrm{d}\tau \right\}, \tag{46b}$$

wherein we have anticipated that κ may be approximated as k at jet noise frequencies: this is justified in terms of the calculations of Schubert (1969) in Ribner (1995, Appendix C). The corresponding result, using the respective Green's function and wave vector $\kappa = k$ (exact in this case) for a virtual medium at rest (Lighthill format), designated by subscript VM, is

$$[\Phi(x, y \mid \omega)]_{VM} = \rho_0^2 k^4 (1/4\pi x)^2 \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{km}(x, \xi_m, \tau) \, \mathrm{e}^{-\mathrm{i}k \cdot \xi_m + \mathrm{i}\bar{\omega}\tau} \xi_m \, \mathrm{d}\tau \right\}. \tag{47}$$

The ratio of these two can be put in the form

$$\Phi(x, y \mid \omega) = [(4\pi x) |G(x, y \mid \omega)|]^2 [\Phi(x, y \mid \omega)]_{VM}. \tag{48}$$

It will simplify discussion if we refer to the factor in brackets as a *normalized Green's function*

$$|G_N(x/x, y \mid \omega)| \equiv |(4\pi x) |G(x, y \mid \omega)|, \tag{49}$$

so that (48) may be written

$$\Phi(x, y \mid \omega) = | |G_N| (x/x, y|\omega)|^2 [\Phi(x, y \mid \omega)]_{VM} \tag{50}$$

with $|G_N|$ dependent solely on the direction of x by virtue of the $4\pi x$ normalization and the $1/x$ decay of $|G|$ in the far field.

4. 'Normalized' Green's function, $|G_N|$

4.1. *Single choice for entire jet*

The 'normalized' Green's function $|G_N|$ defined in (49) is nominally a function of source location. In this section we develop evidence that the value for a single choice of source position y on the axis, designated y_{ref}, may serve as an effective average.

This was concluded in the context of a series of experiments (Atvars *et al.* 1965; Grande 1966) on the far-field directivity pattern of a 'point' source of sound immersed in a subsonic jet. Except for some uncertainty as to the accuracy of simulation of a point source, the measurements effectively yielded values of $|G|^2$ normalized by the value at $\vartheta = 90°$. This result was deemed equivalent to $|G_N|^2$, on the grounds that the effect of wave convection was expected to be nil at 90°; that is, $|G|$ should reduce to $(1/4\pi x)$ there. It was found that the geometric average of the ϑ-dependence for symmetric off-axis positions $+\vartheta$ or $-\vartheta$ differed little from that of the on-axis source position for a given axial location y_1. Further, the variation with y_1 was small. This justifies referring $|G_N|^2$ to a single location, which will greatly simplify both utility and interpretation.

Noting that the direction $\mathbf{x}/x$ may be designated ϑ, φ in polar coordinates for a more general, non-round, jet, the effective average of the squared 'normalized Green function' may be defined as $|G_N|^2$, thus

$$|G_N(\vartheta, \varphi, \mathbf{y}_{ref} \,|\, \omega)|^2 = \langle (4\pi x)^2 |G(\mathbf{x}, \mathbf{y} \,|\, \omega)|^2 \rangle_{av}, \tag{51}$$

where $\mathbf{y}_{ref}$ is a representative value of y_1 along the jet axis. (The dependence on φ, of course, disappears for a round jet.) According to the arguments above, this refers to an average over $\mathbf{y}$. In practice, however, it would be used as a surrogate for a weighted average; that is, equivalent to a single average value of $|G(\mathbf{x}, \mathbf{y} \,|\, \omega)|^2$ taken outside the integral of (40) to replace the value inside that varies with $\mathbf{y}$. The use of a surrogate single Green's function has also been the practice in solutions of the Lilley equation (Mani 1976; Balsa 1976, 1977); Mani referred to the same experiments cited above as justification. Replacement of the $\mathbf{y}$-dependent Green's function of (49) by the single $\mathbf{y}$-independent Green's function of (51) allows immediate integration of (50) into

$$\Phi(\mathbf{x} \,|\, \omega) = |G_N(\vartheta, \varphi, y_0 \,|\, \omega)|^2 \, [\Phi(\mathbf{x} \,|\, \omega)]_{VM}. \tag{52}$$

This states that the same frequency-dependent $|G_N|^2$ that applies to unit volume is, to a sufficient approximation, applicable to the jet as a whole.

The left-hand side of (52) refers to the power spectral density of the jet noise at $\mathbf{x}$ with full allowance for convected wave effects of the mean jet flow. The right-hand-side bracketed term, subscripted VM, is the corresponding power spectral density calculated by Lighthill-based methods that do not allow for the jet flow: this has a multiplier, the squared normalized Green's function $|G_N|^2$. *The approximate relation (52) for the entire jet and the corresponding 'exact' relation (50) for unit volume (invoking (46)) are the central results of this paper.* We turn now to the implications.

4.2. $|G_N|$ *governs wave convection effects*

The form of (52) tells us that the wave convection effects, refraction and shielding, are embodied in the normalized Green function $|G_N|$. This factor squared modifies the smooth directional pattern of intensity otherwise predicted ($|G_N|^2$ taken as unity). This is displayed in the evaluations of $|G_N|^2$: experimental (injected 'point' source: Atvars *et al.* 1965; Grande 1966) and computational (Schubert 1969, 1972*a, b*; Mungur, Plumblee & Doak 1974). The most striking effect is a progressive reduction of intensity within a 'cone of (relative) silence' (figures 1 and 3) opening downstream along the jet axis: this is due to the sound having been refracted outward by the jet velocity gradients.

For filtered jet noise there is a similar 'cone of silence' that depends with frequency. Several examples are compared with corresponding 'point' source measurements in figure 3 (in decibels, labelled 'pure tone'). The cited measurements of $|G_N|^2$ were

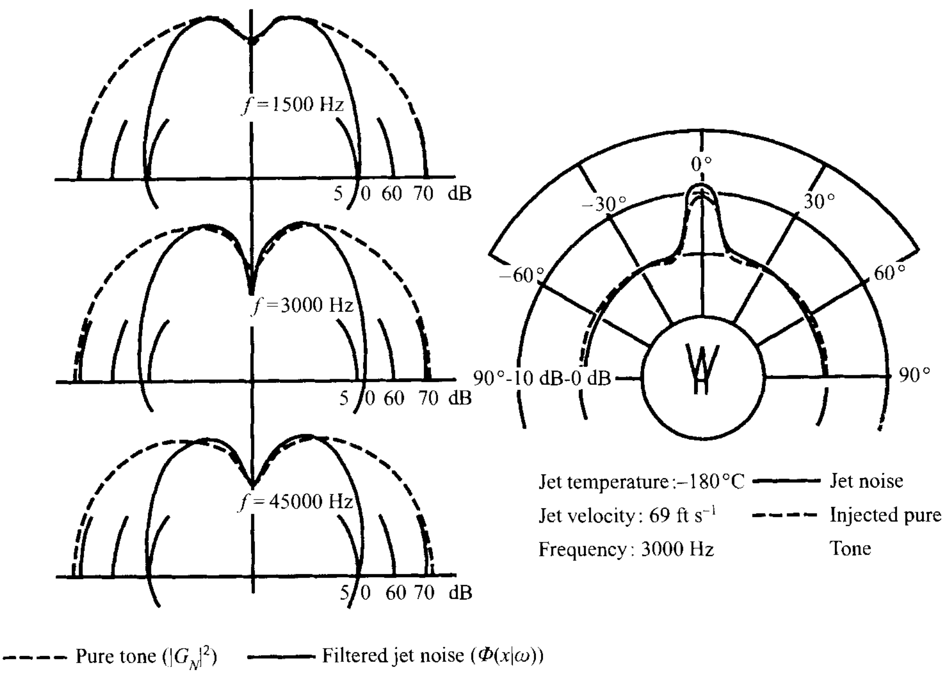

FIGURE 3. Comparison of measured $|G_N|^2$ with narrow-band filtered jet noise to show agreement within a conical region about the axis (refractive zone). Left-hand side: room temperature jet ($M = 0.5$), refractive zone is 'cone of silence' (rays turn outward, reducing axial intensity); right-hand side: very cold jet (-180 °C), refractive zone is enhanced intensity lobe (rays turn inward to a quasi-focus). (After Ribner 1981, based on Atvars *et al.* 1965 and Grande 1966.)

normalized to unity, or zero decibels, at $\vartheta = 90°$, but the curves in figure 3 have been radially shifted (a constant number of dB has been added) for best agreement near the axis. The close match at each frequency is impressive evidence that $|G_N|^2$ dominates the directivity defining the 'cone of silence'.

These facts, applied to (50) and (52), permit the following interpretation: the factor $|G_N|^2$ serves to extend the Lighthill-based calculations into the refractive zone near the jet axis: the 'cone of silence' of figure 1 (but a focused lobe for cryogenic jets, see later). Reversing the dB shift of figure 3 shows that $|G_N|^2$ is near unity (0 dB) outside this zone. Insertion of this value in (52) yields $[\Phi(\boldsymbol{x}\,|\,\omega)]_{VM}$, the Lighthill-based pattern. The value unity seems an adequate approximation except when there are shrouding jets: in that case the deviation from unity should quantify the shielding effect.

The Lighthill-based calculations need not be formulated in terms of the four-dimensional Fourier transform of (46), despite its figuring in the derivation: other formulations deemed to be equivalent may be used. Most are formulated in space–time rather than wavenumber–frequency. In practice, relatively crude simplifying assumptions have been used: one is the 'compact eddy' approximation discussed earlier; another is drastic oversimplification of the functional form (explicit or implicit) of the source correlation function R_{km} of (47) (e.g. a Gaussian form: see later).

Schubert's (1969, 1972*a, b*) evaluation of $|G_N|^2$ used an approach akin to computational fluid dynamics: he obtained a qualitative match with experiment, with fairly good agreement at low frequencies and Mach numbers. However, the computed depth in decibels of the 'refraction valley' was much exaggerated at high frequencies and high subsonic Mach numbers. The later analytical/numerical evaluation by

Munger *et al.* (1974) remedied this deficiency: the computed depth was now within several dB of the experimental values.†

Schubert's studies shows that the frequencies for which geometric acoustics is applicable are many-fold higher than those of jet noise. It was found that at these high frequencies the computed valley depth is grossly exaggerated (e.g. 90 dB prediction at $M = 0.3$). Despite this, some studies (e.g. Csanady 1976 and Morfey *et al.* 1978) attempt to quantify the 'cone of silence' via geometric acoustics.

For heated or cooled jets, or jets of foreign gases, sound-speed gradients come into play (Atvars *et al.* 1965; Grande 1966; Schubert 1969, 1972 *a, b*). Heating enhances the outward refraction, hence increases the depth of the refraction valley. Cooling has the opposite effect. If cold enough, the temperature gradients could dictate inward refraction strong enough to overpower the outward refraction imposed by the velocity gradients. This would give rise to some 'focusing' enhancement of noise intensity along the jet axis: $|G_N|^2$ should exhibit an axial lobe in place of a 'cone of silence'. This expectation was dramatically confirmed in the experiments of Grande: the enhancement lobe was 9 dB at 3000 Hz for an $M = 0.112$ jet of nitrogen at -180 °C. An almost identical lobe was found in his measurements of the jet noise in a narrow filter band at the same frequency (figure 3). Schubert's approximate numerical calculations of $|G_N|^2$ showed a similar, albeit exaggerated, lobe.

5. Discussion

5.1. *Central result*

The central result of the paper may be restated in simplified terms. Lighthill posed his aerodynamic sound sources as radiating into a 'virtual medium at rest.' Refraction of sound (creating the axial 'cone of silence') was suppressed by approximating the density in the dominant source term as constant ($\rho = \rho_0$). But by deferring the step $\rho = \rho_0$, we can pose the radiation as being emitted into the actual jet flow. This brings the refractive effect of the flow gradients into play. Moreover, the residual sound source term is the same.

Mathematically, the only change is replacement of the solution for a pure tone point source in a medium at rest by the solution for the source in a jet flow. The former can be written down by inspection as $\mathrm{e}^{ikr}/4\pi r$; the latter is a complicated solution, $G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)$, of a convected wave equation. But, at frequencies characteristic of jet noise, we find from both experiment and calculation that G reduces in the far field to $\mathrm{e}^{ikr}/4\pi r$ (with a phase shift) times a directional factor. That directional factor, for a single round jet, is near unity for angle ϑ greater than some value ϑ_M. For smaller angles it decreases sharply to a minimum on the jet axis. $\vartheta = 0$. This describes the 'cone of silence' (figures 1 and 3). In summary, in the far field the new G differs in amplitude from the Lighthill $\mathrm{e}^{ikr}/4\pi r$ significantly only within the 'cone of silence'. Use of G thus serves to extend the Lighthill-based solutions into this refractive zone. But outside it may be dispensed with, with little error.

† Mungur *et al.* (1974) model a round jet as though it were contained in a (virtual) conical nozzle of 10° half-angle: the streamlines are along radial lines. This differs from a free jet for which the flow is very close to axial. (Entrainment of the outer fluid modifies this only very near the jet boundary, where the velocity is very small.) Apparently this difference has little effect on the refractive effect of the flow, judging by the good agreement with experiment.

5.2. *Relationship to other approaches*

A variety of Lighthill-based solutions – formalisms for jet noise prediction – has been used with some success (e.g. Krishnappa & Csanady 1969; Ribner 1969; Pao & Lowson 1970; Moon & Zelazny 1975). They were all approximations. As discussed, they usually involved simplistic replacements for the four dimensional Fourier transform formalism. The turbulence correlation function, if it was modelled at all (rather than bypassed by heuristic assumptions), was normally taken as separable in space and time. Chu (1966) improved the model, removing the spurious separability: he used data from his own comprehensive program of very credible measurements by hot wire. He did, however, avoid the four dimensional Fourier transform in the expressions for spectral density by invoking the 'small eddy' assumption. Nevertheless, despite these deemed improvements, his predictive accuracy fell far short of the best in the cited references. Hindsight suggests the capabilities of his data may not have been optimally exploited. A revisit in the light of the present formalism could be profitable.

The predictive problem is compounded by the difficulty of a four-dimensional Fourier transform. As noted, the 'compact eddy' approximation reduces this to a simple volume integral. But, unless corrected by a stratagem, this carries the price of overpredicting the convective amplification. The approach has, indeed, led to fairly accurate predictions, both for round and more complex jet configurations. However, even retaining this approximation, the predictive formalism could be improved as elaborated under (64) of Ribner (1994). The further inclusion of the factor $|G_N|^2$ will extend the solution into the 'cone of silence' (small-θ region), and even improve the accuracy outside this region (by the amount $|G_N|^2$ differs from unity).

Lilley's wave equation, in the hands of Balsa (1977), leads to a similar formula of form $|G_B|^2 \times$ four-dimensional Fourier transform of source correlation function. Here again, in practice the Fourier transform is bypassed by an approximation. There are other important differences. A major component, the shear term included in (7), is missing from the source term (except for a higher-order portion, normally neglected) – the consequences have been discussed; and the squared normalized Green function, $|G_B|^2$, refers to a moving point source. The effects of source convection and wave convection – respectively governing amplification and refraction (via velocity gradients) – are thereby combined. In contradistinction, these effects are decoupled herein by the use of the value of $|G_N|^2$ for a stationary source. That is, $|G_B|^2$ plays essentially the same role as [$|G_N|^2$ times amplification factor]: the two should be largely equivalent.

It is this decoupling that allows the simplicity of the Lighthill-based formalisms to be applied outside the 'cone of silence', since $|G_N|^2$ is near unity there. Another advantage is that this refractive $|G_N|^2$ (stationary source) is determined for a realistic spreading jet – either by calculation or experimentally – whereas $|G_B|^2$ (moving source) has been evaluated only for idealized, infinite, nonspreading jets.

5.3. *Two-microphone cross-correlations*

An allusion to 'fairly accurate predictions' has been made several paragraphs above: it refers to the credibility of Lighthill-based methods for predicting jet noise at angles outside the 'cone of silence'. Even stronger evidence for this credibility is afforded by correlations of two microphones located on a large sphere centred on the jet nozzle: with one microphone fixed, the other is displaced either along a meridian or a circle of latitude. This has the virtue that correlations of two microphones are sensitive to details of the source instantaneous directivity, whereas the single microphone mean

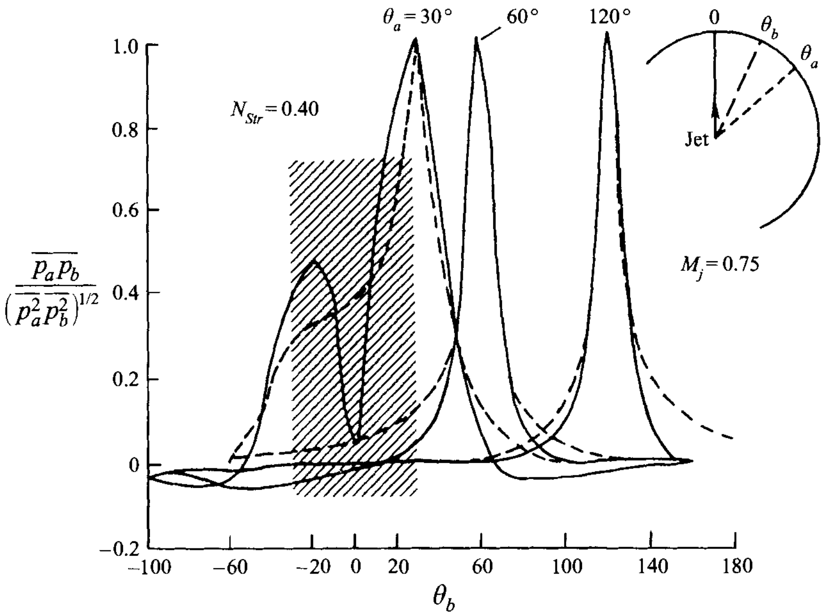

FIGURE 4. Comparison of – – –, Lighthill-based predictive theory with ——, experiment for two-microphone cross-correlations. Such correlations are sensitive to instantaneous directivities of component source radiation patterns. The close agreement beyond the shaded refractive zone ('cone of silence') supports applicability of the theory in the outer region. (After Ribner 1978.)

square response is not. It was with a series of such measurements carried out by Maestrello (1976) that the theory was compared. The Lighthill theory as developed by Ribner (1969) was extended to deal with this case (Ribner 1978: broadband; Richarz 1979: narrow frequency bands). It was found that prediction of two-microphone cross-correlations along circles of latitude showed good qualitative agreement over a range of angular separations, and for different latitudes. This was true both in broadband and the more demanding narrow bands (see also Musafir, Slama & Zindeluk 1984; Musafir 1986). Microphones located along a meridian exhibit a cusp-like correlation in broadband, decaying sharply with separation (Maestrello 1976). The agreement of the theory here was particularly striking outside the 'cone of silence' (figure 4, after Ribner 1978). It makes a strong case for the applicability of the basic Lighthill theory in all the outer region.

5.4. *Issue of convective amplification*

Convection of the jet turbulence by the mean flow is reflected in the motion of the 'source' terms producing the noise. The Doppler effect yields a 'convective amplification' maximizing in the downstream direction (opposed, however, by the refractive 'cone of silence'). This is accounted for in the Lilley theory by use of a moving-source Green's function. The Lighthill theory, as we have used it (Ribner 1964 and ff.) employs a stationary-source Green's function: the amplification is inferred instead from the space–time correlations in the source field. These have the form of a moving pattern and permit a direct calculation. For mathematical simplicity, a Gaussian form of correlation was employed, resulting in an amplification factor for the broadband radiated intensity per unit volume. This displayed the $M \rightarrow 0$ intensity as multiplied by a convective amplification factor, C^{-5}, given by

$$C^{-5} \equiv [(1-M_c\cos\vartheta)^2+\alpha^2 M_c^2]^{-5/2}, \quad \alpha^2 = \omega^2 L^2/\pi c_0^2 \tag{53}$$

in terms of the source pattern convection Mach number M_c; a dependence on the product of frequency times turbulence scale in α, which is only weakly variable, was ignored by setting $\alpha =$ constant.

All of the treatments of convective amplification involve approximations of one kind or another. They figure in the favourable comparisons of theory with experiment cited in the Introduction. For the Lilley-based moving-source Green's function approach, we quote Mani (1976*a*): 'To allow for jet spread, etc., the comparisons are all carried out assuming eddies moving at 65% of the nominal ideal-jet velocity in a slug flow, which is itself assumed to be 65% of the nominal ideal-jet velocity.' Balsa (1976*a*) makes a similar assumption. This may be a judicious approximation, but its quantitative equivalence to the jet scenario is open to question. Thus the good agreement with experimental jet noise directivity outside the 'cone of silence' may be partly fortuitous.

On the other hand, the Lighthill-based convection factor C^{-5}, defined in (53), likewise contains adjustable parameters, most notably the eddy convection Mach number M_c (the other parameter α is relatively weak). Ribner's (1977, 1981) good agreement with the same measurements results primarily from his choice of M_c as 50% of the jet Mach number, rather than Mani's 65%, a value measured by Davies, Fisher & Barratt (1963). So again the agreement may be partly fortuitous. But here also the parameter estimate has some rationale: the Davies *et al.* value is an average along the jet from $y/D = 1.5$ to 4.5, but a better average would cover the major noise generating region, measured with an acoustic mirror technique as $y/D = 4$ to about 8 by Grosche, Jones & Wilhold (1973). This average should come to substantially less than 65% of the jet Mach number, since the jet centreline speed is down to about 82% of the nozzle velocity at $y/D = 8$.

One approximation common to all these approaches is the use of a single convection factor, at each frequency and Mach number, to represent an entire jet. This single factor is, in effect, a weighted average of values tagged to volume elements throughout the jet. This was implicit in the remarks just above. It is not at all clear how much error this might introduce.

5.5. *Issue of shielding*

Mani (1972, 1976*a*, *b*) and Balsa (1977) have pointed to a 'shielding' role of the mean flow in reducing the convective amplification (a function of direction) at high frequencies. But the comparisons of C^{-5} with experiment, cited above, are comparably good without invoking shielding. When we bring in the uncertainties in convective amplification prediction (last paragraph), it would seem there has been no decisive evidence for its reduction by shielding.

We would argue that the frequencies of jet noise are too low for significant shielding: the flow dimensions would have to be much larger than a typical wavelength of the sound (Powell 1960, Ribner 1960). This is also a requirement for geometric acoustics (ray acoustics) to apply. Schubert's calculations (1969, 1972*a*, *b*) of refraction, both by ray acoustics and wave acoustics, show that jet noise is very far from that regime. All these remarks refer to a single round jet.

The case of multitude jets (or equivalent corrugated nozzle jets) is another matter. Substantial shielding of the high-frequency noise of the inner jets by a ring of the outermost jets is a demonstrated fact. Balsa (1976), via his moving-source Green's function, shows apparent agreement with measurements that he cites. No stationary-source Green's function, as proposed herein, has as yet been evaluated for this scenario, but since both techniques incorporate the same physics, there is no reason for such an evaluation to be invalid.

5.6. *Range of applicability*

The results herein are for the far field only. Moreover, they are presumed to be applicable primarily for subsonic jets. For supersonic jets additional noise sources come into play. Tam (1991), in his review article, develops the case for instability waves, identified as 'large scale coherent structures', being a major source of noise. He evaluates the noise directly via a 'stochastic wave model' with very impressive agreement with experiment. We note that these instability waves, to the extent that they coexist with the random turbulence, will contribute to the correlation function R_k used herein: if the contribution were known, the present theory could be applied. Evaluation is another matter: the similarity laws for jet turbulence, which have been used with success for subsonic jets, would have to be re-evaluated for the supersonic regime. A further possible source of error is our replacement of density ρ by its local time average $\bar{\rho}(y)$ in the source terms.

Relevant to the last remark, there will be a pattern of shock waves if the jet does not issue at the design speed from a properly contoured convergent–divergent nozzle. It was shown many years ago that shock–turbulence interaction would generate intense noise (Lighthill 1953, Ribner 1953, 1954). In recent years Tam (summarized in his cited review) has attempted quantitative prediction of this shock-associated noise; he analysed the interaction between instability waves and a 'wave guide' model for the shock structure. His near field patterns show a close match to measurements.

The results are further restricted to jets issuing into ambient fluid at rest: that is, static test conditions. The effects of forward flight on the jet noise are not considered. Michalke & Michel (1979, 1980) have extended the Lighthill theory to provide a successful prediction of these effects. This takes the form of a scaling law that maps the intensity of a static jet at certain jet Mach number and direction into that for a moving jet at an altered Mach number, direction, and distance. (See also an approach via CFD methods (Bayliss *et al.* 1995)). Refraction, governing extension into the 'cone of silence', is not allowed for: this would involve a further development of the present stationary-source Green's function approach.

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Appendix. Generalized convective wave modifications of Lighthill's equation

A.1. *Exact wave equation*

Expansion of Lighthill source term, Q_L

The restriction to a transversely sheared mean flow of uniform density, (3), is relaxed here: Lighthill's source term expression, which we shall call Q_L (the right-hand side of (1)), is expanded under the specifications

$$v_i = U_i + u_i, \quad U_i = U_i(\boldsymbol{x}), \quad \langle v_i \rangle_{av} = U_i, \tag{A 1}$$

$$\rho = \bar{\rho}(x_i) + \rho', \quad \langle \rho \rangle_{av} = \bar{\rho}(x_i). \tag{A 2}$$

By Csanady's (1966) equation (3), Q_L expands as follows:

$$Q_L \equiv \frac{\partial^2 \rho v_i v_j}{\partial x_i, \partial x_j} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 \rho}{\partial t^2} = \rho\left(\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right) + \frac{1}{c_0^2}\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 p}{\partial t^2} - 2v_i\frac{\partial^2 \rho}{\partial x_i \partial t} - v_i v_j \frac{\partial^2 \rho}{\partial x_i \partial x_j}. \quad \text{(A 3)}$$

With this expansion it is easily shown that (1) of the main text is equivalent to Schubert's (1969, 1972) exact wave equation for an inviscid non-heat-conducting fluid. Inserting (A 1) into (A 3) yields, according to Csanady,

$$Q_L = \rho\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i}\frac{\partial U_j}{\partial x_j}\right) - \frac{\partial^2 \rho}{\partial t^2} - 2U_i\frac{\partial^2 \rho}{\partial x_i \partial t} - U_i U_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U_i}{\partial x_j}\frac{\partial \rho u_j}{\partial x_i} + 2\frac{\partial}{\partial x_j}\left(\rho u_j \frac{\partial U_i}{\partial x_i}\right). \quad \text{(A 4)}$$

(When $U_i = (U(x_2), 0, 0)$ this reduces to the right-hand side of (4) of the main text.) With the further expansion

$$\frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U_i}{\partial x_j}\frac{\partial \rho u_j}{\partial x_i} = \rho\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\rho\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + 2u_j\frac{\partial U_i}{\partial x_j}\frac{\partial \rho}{\partial x_i} + 2\frac{\partial u_i u_j}{\partial x_j}\frac{\partial \rho}{\partial x_i} + u_i u_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} \quad \text{(A 5)}$$

and the definition

$$\frac{\hat{D}^2}{Dt^2} \equiv \frac{\partial^2}{\partial t^2} + 2U_i\frac{\partial^2}{\partial x_i \partial t} + U_i U_j \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{(A 6)}$$

and some rearrangement, Q_L becomes

$$Q_L = 2\rho\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\hat{D}^2 \rho}{Dt^2} + \rho\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2u_j\frac{\partial U_i}{\partial x_j}\frac{\partial \rho}{\partial x_i} + 2\frac{\partial u_i u_j}{\partial x_j}\frac{\partial \rho}{\partial x_i} + u_i u_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} + 2\frac{\partial}{\partial x_j}\left(\rho u_j \frac{\partial U_i}{\partial x_i}\right) + \rho\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i}\frac{\partial U_j}{\partial x_j}\right). \quad \text{(A 7)}$$

This expansion of the acoustic source terms, the right-hand side of the wave equation (1), is exact.

A.2. *Approximate wave equations*

Incompressible turbulence

We now introduce approximations in two stages. First, in all but the first three terms of Q_L, we neglect the compressibility of the turbulence in the application to subsonic flows. We argue that density perturbations ρ' in the non-excepted terms account for scattering of sound by turbulence, and they may be neglected in dealing with generation of sound. Thus, in these terms, ρ may be replaced by its local temporal mean $\bar{\rho}(\boldsymbol{x})$. Consistently, it is implied that u_i, in all but the excepted terms, contains no compressible component. Despite this assumed incompressibility of these source terms, sound (pressure waves) will indeed by generated, as Lighthill (1952) showed. Thus compressibility has been retained where acoustically necessary: in the final left-hand-side wave operator.

The first excepted term is $2\rho(\partial U_i/\partial x_j)(\partial u_j/\partial x_i)$. By the argument in the main text above (7), the very small compressible, or acoustic part of this term, being linear in u_j, participates to the first order in wave propagation; thus we move it to the left-hand-side wave operator. The terms $c_0^{-2}\partial^2 p/\partial t^2$ and $-\hat{\mathrm{D}}^2\rho/\mathrm{D}t^2$ have also to do with wave propagation; we move them likewise to the left-hand side, approximating $\hat{\mathrm{D}}^2\rho/\mathrm{D}t^2$ by $\bar{c}^{-2}\hat{\mathrm{D}}^2\mathrm{p}/\mathrm{D}t^2$. (The order of magnitude of the error in this approximation is estimated in Ribner 1995, Appendix B.)

With these term shifts from the right-hand side (Q_L) and approximation of ρ by $\bar{\rho}$ therein, (1) may be rewritten. First we need an approximate equation for mean flow continuity. Taking the correlation $\overline{\rho' u_i}$ as negligible yields this as

$$\partial\bar{\rho}\, U_j/\partial x_j = 0. \tag{A 8}$$

By virtue of (A 8), two terms of the approximate Q_L may be collapsed into one:

$$2\frac{\partial}{\partial x_j}\left(\bar{\rho} u_j \frac{\partial U_i}{\partial x_i}\right) + 2u_j\frac{\partial U_i}{\partial x_j}\frac{\partial \bar{\rho}}{\partial x_i} = -2U_i\frac{\partial}{\partial x_j}\left(u_j\frac{\partial\bar{\rho}}{\partial x_i}\right). \tag{A 9}$$

The modified (1) then reads

$$\begin{aligned}\frac{1}{\bar{c}^2}\left[\frac{\partial^2 p}{\partial t^2} + 2U_i\frac{\partial^2 p}{\partial x_i\,\partial t} + U_i U_j\frac{\partial^2 p}{\partial x_i\,\partial x_j}\right] - 2\bar{\rho}\frac{\partial U_i}{\partial x_j}\frac{\partial u_{jac}}{\partial x_i}&\\ -\nabla^2 p = \bar{\rho}\frac{\partial^2 u_i u_j}{\partial x_i\,\partial x_j} + 2\bar{\rho}\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + 2\frac{\partial u_i u_j}{\partial x_j}\frac{\partial\bar{\rho}}{\partial x_i} + u_i u_j\frac{\partial^2\bar{\rho}}{\partial x_i\,\partial x_j} - 2U_i\frac{\partial}{\partial x_j}\left(u_j\frac{\partial\bar{\rho}}{\partial x_i}\right)&\\ + \bar{\rho}\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i}\frac{\partial U_j}{\partial x_j}\right).&\end{aligned} \tag{A 10}$$

Transversely sheared flow

Let us specialize now to a transversely sheared flow ((3) of the main text), with transverse density gradient as well:

$$U_i = (U(x_2), 0, 0), \quad \bar{\rho} = \bar{\rho}(x_2). \tag{A 11}$$

The modification (A 10) of (1) simplifies to

$$\begin{aligned}\frac{1}{\bar{c}^2}\left[\frac{\partial^2 p}{\partial t^2} + 2U\frac{\partial^2 p}{\partial x_1\,\partial t} + U^2\frac{\partial^2 p}{\partial x_1^2}\right] - 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p&\\ = \bar{\rho}\frac{\partial^2 u_i u_j}{\partial x_i\,\partial x_j} + 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_2}{\partial x_1} + 2\frac{\partial u_2 u_j}{\partial x_j}\frac{\partial\bar{\rho}}{\partial x_2} + u_2^2\frac{\partial^2\bar{\rho}}{\partial x_2^2}.&\end{aligned} \tag{A 12}$$

But it can be quickly verified by direct expansion that

$$\bar{\rho}\frac{\partial^2 v_i v_j}{\partial x_i\,\partial x_j} = \bar{\rho}\frac{\partial^2 u_i u_j}{\partial x_i\,\partial x_j} + 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_2}{\partial x_1}, \tag{A 13}$$

so that an alternative form is

$$\begin{aligned}\frac{1}{\bar{c}^2}\left[\frac{\partial^2 p}{\partial t^2} + 2U\frac{\partial^2 p}{\partial x_1\,\partial t} + U^2\frac{\partial^2 p}{\partial x_1^2}\right] - 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p&\\ = \bar{\rho}\frac{\partial^2 v_i v_j}{\partial x_i\,\partial x_j} + 2\frac{\partial u_2 u_j}{\partial x_j}\frac{\partial\bar{\rho}}{\partial x_2} + u_2^2\frac{\partial^2\bar{\rho}}{\partial x_2^2},&\end{aligned} \tag{A 14}$$

where it is recalled that v_i is the instantaneous resultant flow U_i+u_i, as defined in (A 1). An equation of the form (A 14) results also for a cylindrical jet $U_i = (U(r), 0, 0)$, $\bar{\rho} = \bar{\rho}(r)$, if u_2, x_2 are replaced by u_r, r, respectively (see (A 15)). Equations (A 12) and (A 14) are the key results of this Appendix; they generalize (7) and (10) of the main text, respectively, to the case of flows of non-uniform mean density. The applications and implications are discussed below.

Jet flow

Following Schubert (1969, 1972), we note that U_i in a jet is essentially unidirectional:

$$U_i = [U(r;x_1), 0, 0], \quad r = \sqrt{(x_2^2+x_3^2)}^{\,1/2} \quad \text{round jet}$$
$$= x_2 \quad \text{two-dimensional jet,} \qquad \text{(A 15)}$$

with a strong dependence on r, and a weak dependence on x_1. Further,

$$\bar{\rho} = \bar{\rho}(r;x_1), \qquad \text{(A 16)}$$

with a similar dependence. Thus, for both U and $\bar{\rho}$, the gradients along x_1 are very much less than those along r. For the foregoing transversely sheared flow the x_1-gradients are identically zero; requiring this led to (A 12)–(A 14), as well as (7) and (10). It follows that (A 12) to (A 14) may be applied to jet flow as a close approximation, using (A 15) and (A 16), with x_2 replaced by r. With this interpretation, the Green's function $G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)$ calculated throughout this paper may refer to a realistic spreading jet. On the other hand, the $G(\boldsymbol{x}, \boldsymbol{y} \mid \omega)$ obtained in the Lilley-based procedures refers to a jet modelled as a non-spreading infinite cylinder

Density scenarios: $\bar{\rho}$ = *constant vs.* $\bar{\rho} = \bar{\rho}(r)$

Let us specialize further to a uniform mean density,

$$\bar{\rho} = \rho_0 = \text{constant.} \qquad \text{(A 17)}$$

This, together with (A 11), recovers the scenario of the main text. It is seen that the density gradient terms drop out, and (A 12) and (A 14) reduce to (7) and (10), respectively: the modified Lighthill equation in the form of (10) is confirmed as a special case of the more general form of this Appendix, (A 14).

From the foregoing, it is clear that density gradients, via the additional source terms, cause more noise to be generated. This has been explored in the context of hot jets by Morfey (1973), Mani (1976*b*), and Michalke & Michel (1979, 1980). The source terms in (A 12), (A 14) appear similar to those deduced by Mani. The term $2(\partial u_2 u_j/\partial x_j)(\partial\bar{\rho}/\partial x_2)$ is essentially of dipole form, $\partial Q_j/\partial x_j$, (treating $\partial\bar{\rho}/\partial x_2$ as a spatial constant): it would yield a factor κ^2 in place of κ^4 in an equation like (40). As a consequence of this, or by arguments given in the cited references, the corresponding radiated sound power would vary as U^6; they showed it could exceed the ordinary quadrupole-source jet noise, with its U^8 law, for sufficiently hot jets. The term $u_2^2\,\partial^2\bar{\rho}/\partial x_2^2$ is of monopole form, leading to a U^4 law (κ^4 factor replaced by unity). This would radiate very weakly, the curvature $\partial^2\bar{\rho}/\partial x_2^2$ being minimal in the zone of strongest turbulence, where the mean flow shear and $\partial\bar{\rho}/\partial x_2$ both maximize.

Over time '...dozens of equivalent (and nonequivalent) source term expansions have been published by flow noise researchers. This multiplicity of competing source terms has been a major contributor to confusion...' (Ribner 1981). In this author's view, the effective limit in the law of diminishing returns has been reached in the expansions of this Appendix: of the Lighthill wave operator (left-hand side of (A 10)) and of the Lighthill source term for an inviscid non-heatconducting fluid (exact, (A 7); approximate (A 14)).

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